Applications of Linear Systems
Reading: Lay 1.6

September 9, 2013

This section is a nice break from theory. We have already covered enough material to have some interesting applications to problems.

In a physical or social science (physics, engineering, economics, etc), you generally are studying a real-world system with complicated behavior and want to find a way to predict the behavior of the system. For instance, you might want to predict the amount of traffic on different city streets tomorrow afternoon. Often, you make such predictions by writing down a "mathematical model": some set of equations which is complicated enough to make good predictions but still simple enough to solve.

In today's lecture, we describe how linear systems arise as mathematical models in various contexts, and we show how the techniques we learned in the last few classes can be used to make useful predictions.

1 Chemical Equations

One simple application of linear equations is in the study of chemical reactions. A particular molecule has a chemical formula, describing the atoms it is made of. For instance, water has the chemical formula H_2O: one molecule of water is made out of two atoms of hydrogen and one atom of oxygen. Similarly, the table sugar that I put in my morning coffee is made out of molecules of sucrose, with chemical formula $C_{12}H_{22}O_{11}$; each molecule has 12 atoms of carbon, 22 of hydrogen, and 11 of oxygen.

In a chemical reaction, you start with molecules of one or more substances, which undergo a transformation into molecules of one or more possibly different substances. In such a reaction, atoms cannot be created or destroyed.

In this way, there is a “balance” in the chemical reaction; if I start with a certain number of atoms of hydrogen, for instance, I must end with the same number of atoms of hydrogen.

Example 1.1. We are operating a factory which produces hydrogen (chemical formula H_2) and oxygen (chemical formula O_2) gas by running an electrical current through water. This process is called electrolysis, and proceeds according to an equation of the form

$$x_1\, H_2O \longrightarrow x_2\, H_2 + x_3 O_2. \tag{1}$$

What are the possible values of $x_1.x_2, x_3$ such that the equation is balanced– that is, such that no oxygen or hydrogen is created or destroyed?

Because the atoms of each individual element (hydrogen, oxygen) are counted separately, we must keep track of them separately. We keep track of atoms by using vectors. The different entries in each vector correspond to the different atoms in the reaction. So, for instance, we could choose to keep track of the hydrogen atoms using the first entry of our vectors and the oxygen atoms using the second entry of our vectors. In this case, we could write

$$1H_2O = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad 5\, H_2O = \begin{bmatrix} 10 \\ 5 \end{bmatrix},$$

and so on. In this notation, the equation (1) is balanced if and only if

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{2}$$

Rewriting (2) by moving all terms to the left-hand side gives

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} - x_3 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or in matrix form}$$

$$\begin{bmatrix} 2 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

The equation (3) is a homogeneous equation of the form $A\mathbf{x} = \mathbf{0}$. We can find an expression for the possible values of x_1, x_2 that balance the reaction by writing down the augmented matrix:

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix},$$

and then solving for the RREF, which turns out to be

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}. \tag{4}$$

The RREF in (4) corresponds to the linear system

$$\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 - 2x_3 &= 0, \end{aligned}$$

which has one free variable x_3. Thus, the chemical reaction is balanced when it has the form

$$2x_3\, H_2O \longrightarrow 2x_3\, H_2 + x_3 O_2, \tag{5}$$

for any choice of coefficient x_3. For instance, choosing $x_3 = 5$ shows that

$$10\, H_2O \longrightarrow 10\, H_2 + 5O_2$$

is balanced; we get out the same amount of hydrogen and oxygen that we put in. It is the usual convention among chemists to write a balanced chemical equation using the smallest whole numbers possible, because this is usually easiest to read. This means that we choose any free variables in order to give us the smallest whole number coefficients possible. Using this convention, (5) becomes

$$2H_2O \longrightarrow 2H_2 + O_2.$$

Example 1.2. Sulfuric acid is made of hydrogen, sulfur, and oxygen atoms (formula H_2SO_4). It is commonly produced in a reaction from water (H_2O) and sulfur trioxide (SO_3) by a reaction of the form

$$x_1\, H_2O + x_2\, SO_3 \longrightarrow x_3\, H_2SO_4\ . \tag{6}$$

What is the balanced form of (6)? Rewriting as a vector equation, where the first entry of the vectors corresponds to hydrogen, the second to sulfur, and the third to oxygen, gives:

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - x_3 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Again we write the augmented matrix

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & -4 & 0 \end{bmatrix},$$

and we solve for the RREF. The resulting RREF is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a linear system with x_3 as the only free variable; solving for the free variable gives $x_1 = x_3,\ x_2 = x_3$. So the equation is balanced as long as it has the form

$$x_3\, H_2O + x_3\, SO_3 \longrightarrow x_3\, H_2SO_4\ . \tag{7}$$

Again putting (7) in a form with lowest whole number coefficients, we see that the standard form of the balanced equation is

$$H_2O + SO_3 \longrightarrow\ H_2SO_4\ .$$

2 Economic models

Here's a simple model of a nation's economy. Imagine the economy is divided into "sectors": groups of companies which produce the same thing. For example, the economy might have a sector of companies that produce wheat, and a sector of companies that produce steel, and so on. The output of a given sector is divided among other sectors of the economy–for instance, some of the steel sector's output might be used by the wheat sector.

Suppose that for every sector of the economy we know exactly how the output of the sector is divided among the other sectors of the economy. Call the total yearly dollar output of a sector the "price" of that sector. Then we can find **equilibrium prices**, values of the price for each sector of the economy so that the price of a given sector exactly balances its "expenses", the dollar amount of output it uses from other sectors of the economy. Note that a sector can have expenses from itself–that is, it can be using some of its own output.

Example 2.1. Suppose a nation's economy is divided into three sectors: Coal, Oil, and Steel. Say that the total output of each sector is divided as follows:

- The coal sector buys 20% of its own output, 50% of the oil sector's output, and 20% of the steel sector's output.
- The oil sector buys 30% of the coal sector's output, 10% of its own output, and 70% of the steel sector's output.
- The steel sector buys 50% of the coal sector's output, 40% of the oil sector's output, and 10% of its own output.

Writing these as equations, where p_C, p_O, and p_S are the prices of the coal, oil, and steel sectors respectively, gives

$$\begin{aligned} p_C &= .2p_C + .5p_O + .2p_S \\ p_O &= .3p_C + .1p_O + .7p_S \\ p_S &= .5p_C + .4p_O. + .1p_S \end{aligned}$$

Moving everything to the left-hand side, these equations become

$$\begin{aligned} .8p_C - .5p_O - .2p_S &= 0 \\ -.3p_C + .9p_O - .7p_S &= 0 \\ -.5p_C - .4p_O + .9p_S &= 0. \end{aligned}$$

Writing the augmented matrix gives

$$\begin{bmatrix} .8 & -.5 & -.2 & 0 \\ -.3 & .9 & -.7 & 0 \\ -.5 & -.4 & .9 & 0 \end{bmatrix} \tag{8}$$

We compute the RREF of (8) using our row reduction algorithm:

$$\begin{bmatrix} 1 & 0 & -53/57 & 0 \\ 0 & 1 & -62/57 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the system is solved by any vector of the form

$$\begin{bmatrix} (53/57)p_S \\ (62/57)p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} 53/57 \\ 62/57 \\ 1 \end{bmatrix}. \tag{9}$$

That is, the equilibrium prices are any multiple of the vector whose entries are 53/57, 62/57, 1. You can think of this as saying that the coal sector is 53/57 times as big as the steel sector, etc.

3 Network Flow

Say we have a grid of water pipes. We depict it with something called a "network", a diagram which consists of "branches" which represent pipes and "junctions" or "nodes" that represent points where the branches connect. Each branch carries a "flow", or amount of water. The rules of networks are:

- The total flow into a junction equals the total flow out of the junction;
- The total flow into the network equals the total flow out of the network.

If either rule were disobeyed, water would be building up somewhere and the pipes would burst.

We usually depict a network by labeling flows in each branch next to the branches. Some flows are generally given to you as numbers, and some are variables. By applying the two rules above, we get a linear system which contains one equation for each node, plus one equation for the system as a whole.

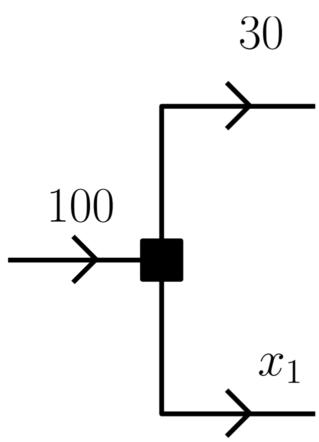

Figure 1: A simple network.

Consider the network in Figure 1.We have one node, so the network rules give us two equations. They are

$$100 = 30 + x_1,$$
$$100 = 30 + x_1.$$

In this case, both rules give us the same equation. But it is OK, we can still solve to see that the flow $x_1 = 70$.

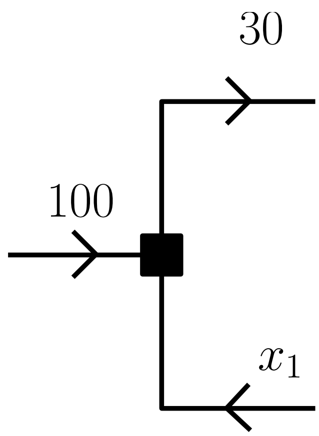

Figure 2: The network from before, modified.

If we instead drew the network as in Figure 2, our equations would be

$$100 + x_1 = 30,$$
$$100 + x_1 = 30.$$

The solution to this sytem is $x_1 = -70$. What this means in terms of the flows is that the water is flowing *opposite* the direction of the arrow. So this agrees with our previous answer.

Figure 3 gives a depiction of a more complicated network. Note that there are 4 nodes, so we will have 5 equations. They are:

$$\begin{aligned}
300 &= x_2 + x_3 \\
200 + x_2 &= x_1 + x_5 \\
300 + x_3 &= 100 + x_4 \\
x_1 + x_4 &= 300 \\
800 &= 400 + x_5
\end{aligned}$$

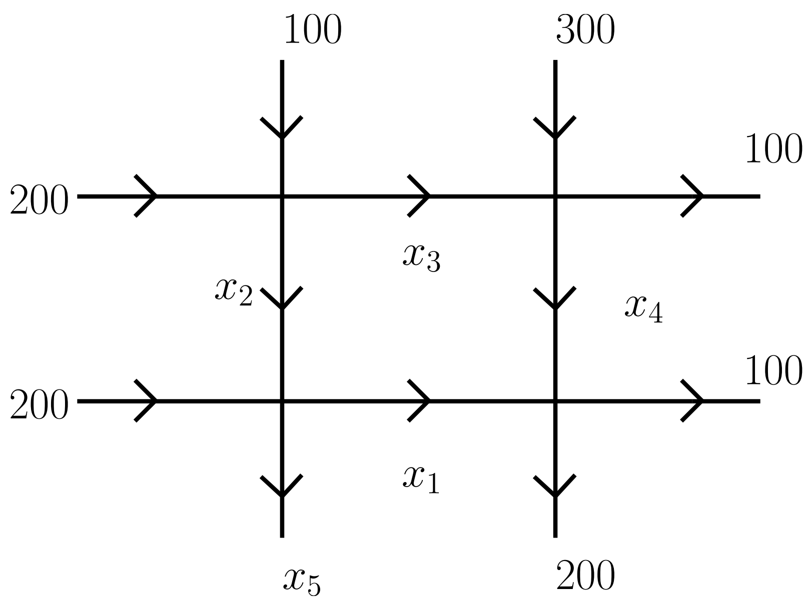

Figure 3: A more complicated network.

Moving all the variables to one side and then writing the augmented matrix gives:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 300 \\ 1 & -1 & 0 & 0 & 1 & 200 \\ 0 & 0 & -1 & 1 & 0 & 200 \\ 1 & 0 & 0 & 1 & 0 & 300 \\ 0 & 0 & 0 & 0 & 1 & 400 \end{bmatrix}.$$

Believe it or not, the RREF of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 300 \\ 0 & 1 & 0 & 1 & 0 & 500 \\ 0 & 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 0 & 0 & 1 & 400 \end{bmatrix}.$$

So there are infinitely many solutions. Here are two:

$$x_1 = 100,\ x_2 = 300,\ x_3 = 0,\ x_4 = 200,\ x_5 = 400$$

$$x_1 = 50,\ x_2 = 250,\ x_3 = 50,\ x_4 = 250,\ x_5 = 400$$

A little thought will explain why infinitely many solutions is a sensible answer(this could be somewhat challenging, but it is an interesting thought experiment!).